A Σ^1_4 wellorder of the reals with NS_{ω_1} saturated

Sy-David Friedman and Stefan Hoffelner [*]

March 11, 2021

Abstract

We show that, assuming the existence of the canonical inner model with one Woodin cardinal M_1, there is a model of ZFC in which the nonstationary ideal on ω_1 is $\aleph_2$-saturated and whose reals admit a Σ^1_4-wellorder.

1 Preliminaries

1.1 Introduction

The investigation of the nonstationary ideal on a regular cardinal has a long history, being strongly tied to the development of several central concepts of modern set theory such as generic ultrapower constructions, Martin's Maximum MM, Woodin's $\mathbb{P}_{max}$-forcing, the stationary tower and many more. The question regarding the length of antichains of stationary subsets modulo nonstationarity, first posed by A. Tarski, generated particular interest as it became clear over time that its answer relies on large cardinals and has deep and surprising effects on the surrounding set theoretic universe. We start with defining the central notion:

Definition 1. *Let κ be a regular, uncountable cardinal and NS_κ the ideal of nonstationary subsets of κ. For a regular cardinal λ we say that NS_κ is λ-saturated if there are no antichains of length λ in $P(\kappa)/NS_\kappa$, where antichains are meant to be modulo NS_κ-small intersections of their elements.*

An equivalent way of saying that NS_κ is λ-saturated is therefore the statement that the Boolean algebra $P(\kappa)/\mathrm{NS}_\kappa$ has the λ-cc, which highlights the importance of the notion in the context of generic ultrapowers where conditions are stationary sets ordered by the subset relation.

There is a long list of research which has been devoted to studying the possible lengths of antichains in $P(\kappa)/\mathrm{NS}_\kappa$, involving many prominent set-theorists. In culminating work M. Gitik and S. Shelah in [3] proved that

---

[*] Keywords: nonstationary ideal, projective wellorders, saturation, AMS Subject Code Classification: 03E35, 03E45, 03E47,03E55.

NS_κ can never be κ^+-saturated for $\kappa > \aleph_1$. The situation for $\kappa = \omega_1$ behaves differently though. It was known since the early seventies from the work of K. Kunen (see [6]) that there can be $\aleph_2$-saturated ideals on ω_1 in the presence of a huge cardinal. Focusing on the nonstationary ideal, the problem was investigated from a different angle using completely different methods by J. Steel and R. Van Wesep who forced over a model of a stronger version of AD to obtain a model of choice where NS_{ω_1} is saturated. In a different line of research, again, the result was later improved with the discovery of Martin's Maximum MM, known to be consistent from a supercompact cardinal, which outright implies that NS_{ω_1} is $\aleph_2$-saturated. The ultimate solution to the problem of the consistency of the statement "NS_{ω_1} is $\aleph_2$-saturated" from optimal large cardinal assumptions was eventually found by S. Shelah who showed in the early 80's that already a Woodin cardinal suffices. In 2006, R. Jensen and J. Steel [5] proved that the assumption of a Woodin cardinal is in fact sharp in terms of consistency strength.

There is a deep and surprising connection between the statement "NS_{ω_1} is saturated" and the Continuum Hypothesis CH. Woodin, improving the earlier mentioned result of Steel and Van Wesep, was able to show that given a measurable cardinal, the saturation of NS_{ω_1} implies a projective failure of CH (see [11], Theorem 3.17). Definable wellorders of the reals enter the picture via a result of G. Hjorth (see [4]), who showed that in the presence of "every real has a sharp," a Σ^1_3-wellorder of the reals implies CH.

The goal of our article will be to construct a model where NS_{ω_1} is saturated and whose reals admit a Σ^1_4-wellorder. In the light of the above mentioned results the Σ^1_4-definable wellorder we obtain is optimal in the presence of a measurable cardinal. Put into a more general context, this work can be seen as an attempt to find new coding methods which work at the level of inner models for Woodin cardinals.

1.2 Some of the notions used

We start to introduce the main notions we will use throughout the proof.

Definition 2. *A cardinal Λ is a Woodin cardinal if for every function $f : \Lambda \to \Lambda$ there is a $\kappa < \Lambda$ with $f"\kappa \subset \kappa$, and an elementary embedding $j : V \to M$ with critical point κ such that $V_{j(f)(\kappa)} \subset M$.*

Definition 3. *Let A be an arbitrary set then a cardinal κ is A-strong up to the cardinal Λ iff $\forall \gamma < \Lambda \, \exists j : V \to M$ which is elementary such that*

1. *$crit\ j = \kappa \wedge \gamma < j(\kappa)$,*
2. *$V_{\kappa+\gamma} \subset M$,*
3. *$A \cap V_{\kappa+\gamma} = j(A) \cap V_{\kappa+\gamma}$.*

We will use the following characterization of a Woodin cardinal.

Fact 4. *The following are equivalent*

- Λ *is Woodin*
- *For any* $A \subset V_\Lambda$, $\{\alpha < \Lambda : \alpha$ *is* A*-strong up to* $\Lambda\}$ *is stationary in* Λ.

We will need a bit more, namely a Woodin cardinal with a $\diamondsuit$-sequence living below it:

Definition 5. *Let* Λ *be a Woodin cardinal then we say that* Λ *is Woodin with* $\diamondsuit$ *iff there is a sequence* $(a_\kappa : \kappa < \Lambda)$ *such that for each* κ, $a_\kappa \subset V_\kappa$ *and for every* $A \subset V_\Lambda$ *the set*

$$\{\kappa < \Lambda : A \cap V_\kappa = a_\kappa \wedge \kappa \text{ is } A\text{-strong up to } \Lambda\}$$

is stationary in Λ.

In terms of consistency strength this adds nothing to being a Woodin cardinal. If we start with an arbitrary ground model V with a Woodin cardinal Λ, then it is known (see [8], Lemma 0.3), that forcing with Λ-Cohen forcing will produce a generic extension of V in which Λ is Woodin with $\diamondsuit$. The classical argument which produces a $\diamondsuit$-sequence in L can be applied to show that in canonical inner models of large cardinals, if Λ is Woodin in such an inner model, then Λ is in fact Woodin with $\diamondsuit$ in that model (see [8], Lemma 0.2). Indeed, these models satisfy a sufficient amount of condensation such that the original proof of Jensen applies.

Fact 6. *Assume that* M *is an inner model of the form* $L[\vec{E}]$, *where* $\vec{E}$ *is a fine extender sequence, which contains a Woodin cardinal* Λ. *Then* Λ *is Woodin with* $\diamondsuit$ *in* M.

Next, we briefly discuss the central notion of forcing which is used to bound lengths of antichains in $P(\omega_1)/\mathrm{NS}_{\omega_1}$. Assume that $\vec{S} = (S_i : i < \kappa)$ is a maximal antichain in $P(\omega_1)/\mathrm{NS}_{\omega_1}$ and we want to pass to a suitable generic extension where $\vec{S}$ has size $\aleph_1$. The naive approach would be to simply collapse κ to $\aleph_1$ but the drawback is, that in the resulting generic extension, $\vec{S}$ will lose its maximality, rendering any iterative argument pointless.

Consequentially in order to show that NS_{ω_1} can be $\aleph_2$-saturated, one needs a way to bound the lengths of antichains in $P(\omega_1)/\mathrm{NS}_{\omega_1}$, yet preserve maximality of antichains in $P(\omega_1)/\mathrm{NS}_{\omega_1}$.

Definition 7. *Assume that* $\vec{S}$ *is an antichain of stationary subsets of* ω_1. *Then the so-called sealing forcing* $\mathbb{S}(\vec{S})$ *consists of conditions of the form* (p, c) *where* $p : \alpha + 1 \to \vec{S}$ *is a function and* $c : \alpha + 1 \to \omega_1$ *is a function with closed image and such that*

$$\forall \xi \leq \alpha (c(\xi) \in \bigcup_{i \in \xi} p(i))$$

holds. We let $(q, d) < (p, c)$ *if* q *and* d *end-extend* p *and* c *respectively.*

It is well known that the sealing forcing $\mathbb{S}(\vec{S})$ is ω-distributive and preserves all stationary subsets of elements $\vec{S}$, i.e., if $S_i \in \vec{S}$ and $T \subset S_i$ is stationary, then T remains stationary in the generic extension by $\mathbb{S}(\vec{S})$. Consequentially $\mathbb{S}(\vec{S})$ is stationary subsets of ω_1 preserving if $\vec{S}$ is maximal. In accordance with standard terminology we will say from now on that a forcing notion $\mathbb{P}$ preserves stationary sets whenever we actually mean that $\mathbb{P}$ preserves stationary subsets of ω_1. With the sealing forcing available, the natural approach to produce a generic extension in which NS_{ω_1} is saturated is to seal off all the antichains in $P(\omega_1)/\mathrm{NS}_{\omega_1}$ iteratively. A maximal antichain, once sealed off remains maximal in all stationary set preserving outer models, as can be easily seen using the generically added club we shot through the diagonal union of elements of the antichain. Thus once we seal off one maximal antichain, its length becomes ω_1 and we have made progress in our attempt of finding a model for NS_{ω_1} saturated.

Knowing what to do in successor stages, we still need to iterate these forcings in a stationary set preserving way. Shelah was able to get around this problem as follows. He introduced a weaker form of properness, namely, semiproperness and found a more general form of the usual countable support iteration, the so-called revised countable support iteration which can be used to preserve semiproperness. A partial order $\mathbb{P}$ is said to be semiproper if and only if there is a cardinal $\theta > 2^{|\mathbb{P}|}$ and there is a club $C \subset [H_\theta]^\omega$ of elementary submodels $M \prec (H_\theta, \in, <, \dots)$ such that every condition $p \in \mathbb{P} \cap M$ has an $(M, \mathbb{P})$-semigeneric condition q below it; and a condition q is $(M, \mathbb{P})$-semigeneric if and only if whenever $\dot{\alpha}$ is a name for a countable ordinal in M then $q \Vdash \dot{\alpha} \in M$. Note that a semiproper notion of forcing preserves stationary subsets of ω_1.

Definition 8. *Let $(\mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta)_{\beta<\alpha}$ be an iteration, α a limit ordinal. Then, $\mathbb{P}_\alpha$ is an RCS-limit (short for revised countable support) of $\mathbb{P}_\beta$, $\beta < \alpha$ if it is a subset of the inverse limit of the forcings $(\mathbb{P}_\beta : \beta < \alpha)$ such that each $p \in \mathbb{P}_\alpha$ satisfies*

> *for each $q < p$ there is an ordinal $\gamma < \alpha$ and a $\mathbb{P}_\gamma$-condition r such that $r \leq q \restriction \gamma$ and in the forcing $\mathbb{P}_\gamma$ it holds that $r \Vdash_\gamma cf(\alpha) = \omega$ or for each $\beta \geq \gamma$ $p \restriction [\gamma, \beta) \Vdash_{\mathbb{P}_{\gamma,\beta}} p(\beta) = 1$.*

The following theorem justifies the added complications in the definition of RCS-iterations (see [7],Theorems 5 and 17).

Fact 9. *Iterations with RCS-support whose factors are semiproper result in a semiproper forcing notion. Moreover, if we split an RCS iteration into two pieces, then the tail iteration, as seen from the intermediate model, will look like an RCS iteration again. More precisely, if $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \leq \beta)$ is an RCS iteration and if $\dot{\mathbb{P}}_{\gamma\beta}$ denotes the factor forcing of $\mathbb{P}_\beta$ over the model $V^{\mathbb{P}_\gamma}$, then $1 \Vdash_\gamma \backslash \dot{\mathbb{P}}_{\gamma\beta}$ is an RCS-iteration," for every $\gamma < \beta$.*

Leaving out almost all the details, Shelah's proof for making NS_{ω_1} saturated from a Woodin cardinal then proceeds as follows: we let Λ be a Woodin cardinal, fix some bookkeeping device to list the maximal antichains in $P(\omega_1)/\mathrm{NS}_{\omega_1}$ and start to seal them off, provided the sealing forcing is semiproper. Taking revised countable support guarantees that this forcing is semiproper, hence stationary set preserving. We iterate Λ-many times and the Woodin cardinal is used to show that in the end no long antichain has survived. A detailed proof of this will be given at the end of this article. We shall say however that Shelah's argument allows some alterations, i.e., we can force with additional posets during the iteration, as long as the forcings used are semiproper and the stages where we seal off maximal antichains in $P(\omega_1)/\mathrm{NS}_{\omega_1}$ remains stationary below the Woodin cardinal.

1.3 $M_1^{\#}$ and M_1

We quickly introduce a couple of properties of M_1, the canonical inner model with one Woodin cardinal, which will serve as the ground model for our forcing construction.

$M_1^{\#}$ denotes as always the least countable mouse which is not 1-small, i.e., there is a λ which is the critical point of an extender on the $\mathcal{M}$-sequence and a $\kappa < \lambda$ such that $\mathcal{J}_\lambda^{\mathcal{M}} \models \kappa$ is a Woodin cardinal . M_1 is the result of iterating away the last extender, hence M_1 is a class sized model with one Woodin cardinal.

J. Steel in [9] showed that for M_1 there is a weaker variant $\mathcal{I}(\mathcal{M})$ of the usual iteration game played on a premouse $\mathcal{M}$ which still ensures a sufficient amount of comparison. We say that a premouse $\mathcal{M}$ is Π^1_2-iterable if player II has a winning strategy for $\mathcal{I}(\mathcal{M})$. As the notation suggests, the set of countable premice which are Π^1_2-iterable is Π^1_2-definable itself (see [10], Lemma 1.7). The winning strategy for II for $\mathcal{I}(\mathcal{M})$ guarantees that $\mathcal{M}$ can be compared with any countable premouse which is an initial segment of M_1, on the other hand, premice $\mathcal{N}$ which are embeddable into initial segments of M_1 will hand player II a winning strategy in the iteration game $\mathcal{I}(\mathcal{N})$. This implies that a nice definition of a cofinal set of countable initial segments of M_1 exists in ω_1-preserving forcing extensions $M_1[G]$ of M_1: we can consider the set B of countable premice which are Π^1_2-iterable, ω-sound and which project to ω. If we consider in $M_1[G]$ an element $\mathcal{M}$ of B and assume it would not be fully iterable, then one can show that in fact $\mathcal{M}$ would have to contain all the reals of M_1. But as $\mathcal{M}$ was assumed to be countable, this contradicts the fact that $M_1[G]$ is an ω_1-preserving extension of M_1. Hence $\mathcal{M}$ must be fully iterable and we can compare it with some $\mathcal{N} = \mathcal{J}_\eta^{M_1}$, $\eta < \omega_1$ an ω-projecting initial segment of M_1. As both models $\mathcal{M}$ and $\mathcal{N}$ are ω-sound and ω-projecting, they actually do not move during the iteration and therefore we obtain that $\mathcal{M} \trianglelefteq \mathcal{N}$ or $\mathcal{N} \trianglelefteq \mathcal{M}$ must hold. If we let the height of $\mathcal{N}$ increase we see that certainly an $\eta < \omega_1$ exists such that $\mathcal{M} \trianglelefteq \mathcal{N} = \mathcal{J}_\eta^{M_1}$ holds. Thus the following is true:

Proposition 10. *Let $M_1[G]$ be an ω_1-preserving forcing extension of M_1. Then in $M_1[G]$ there is Π^1_2-definable set $\mathcal{I}$ of premice which are of the form $\mathcal{J}_\eta^{M_1}$ for some $\eta < \omega_1$. $\mathcal{I}$ is defined as*

$$\mathcal{I} := \{\mathcal{M} \textit{ ctbl premouse} : \mathcal{M} \textit{ is } \Pi^1_2\textit{-iterable, } \omega\textit{-sound and projects to } \omega\},$$

and the set

$$\{\eta < \omega_1 : \exists \mathcal{N} \in \mathcal{I}(\mathcal{N} = \mathcal{J}_\eta^{M_1})\}$$

is cofinal in ω_1.

1.4 Coding reals by triples of ordinals

We present a coding method invented by A. Caicedo and B. Velickovic which we will use in the argument. All results in this section are due to them (see [1]).

Definition 11. *A $\vec{C}$-sequence, or a ladder system, is a sequence $(C_\alpha : \alpha \in \omega_1, \alpha$ a limit ordinal $)$, such that for every α, $C_\alpha \subset \alpha$ is cofinal and the order type of C_α is ω.*

For three subsets $x, y, z \subset \omega$ we can consider the oscillation function. First, turn the set x into an equivalence relation $\sim_x$, defined on the set $\omega - x$ as follows: for natural numbers in the complement of x satisfying $n \leq m$, let $n \sim_x m$ if and only if $[n, m] \cap x = \emptyset$. This enables us to define:

Definition 12. *For a triple of subset of the natural numbers (x, y, z) list the intervals $(I_n : n \in k \leq \omega)$ of equivalence classes of $\sim_x$ which have nonempty intersection with both y and z. Then, the oscillation map $o(x, y, z) : k \to 2$ is defined to be the function satisfying*

$$o(x,y,z)(n) = \begin{cases} 0 & \textit{if } min(I_n \cap y) \leq min(I_n \cap z) \\ 1 & \textit{else.} \end{cases}$$

Next, we want to define how suitable countable subsets of ordinals can be used to code reals. For that suppose that $\omega_1 < \beta < \gamma < \delta$ are fixed limit ordinals, and that $N \subset M$ are countable subsets of δ. Assume further that $\{\omega_1, \beta, \gamma\} \subset N$ and that for every $\eta \in \{\omega_1, \beta, \gamma\}$, $M \cap \eta$ is a limit ordinal and $N \cap \eta < M \cap \eta$. We can use (N, M) to code a finite binary string. Namely, let $\bar{M}$ denote the transitive collapse of M, let $\pi : M \to \bar{M}$ be the collapsing map and let $\alpha_M := \pi(\omega_1)$, $\beta_M := \pi(\beta)$, $\gamma_M := \pi(\gamma)$ $\delta_M := \bar{M}$. These are all countable limit ordinals. Furthermore set $\alpha_N := sup(\pi``(\omega_1 \cap N))$ and let the height $n(N, M)$ of α_N in α_M be the natural number defined by

$$n(N, M) := card(\alpha_N \cap C_{\alpha_M}),$$

where C_{α_M} is an element of our previously fixed ladder system. As $n(N, M)$ will appear quite often in the following we write shortly n for $n(N, M)$. Note that as the order type of each C_α is ω, and as $N \cap \omega_1$ is bounded below $M \cap \omega_1$, n is indeed a natural number. Now, we can assign to the pair (N, M) a triple (x, y, z) of finite subsets of natural numbers as follows:

$$x := \{card(\pi(\xi) \cap C_{\beta_M}) \,:\, \xi \in \beta \cap N\}.$$

Note that x again is finite as $\pi"(\beta \cap N)$ is bounded in the cofinal in β_M-set C_{β_M}, which has ordertype ω. Similarly we define

$$y := \{card(\pi(\xi) \cap C_{\gamma_M}) \,:\, \xi \in \gamma \cap N\}$$

and

$$z := \{card(\pi(\xi) \cap C_{\delta_M} \,:\, \xi \in \delta \cap N\}.$$

Again, it is easily seen that these sets are finite subsets of the natural numbers. We can look at the oscillation $o(x \backslash n, y \backslash n, z \backslash n)$ and if the oscillation function at these points has a domain bigger or equal to n then we write

$$s_{\beta,\gamma,\delta}(N, M) := \begin{cases} o(x \backslash n, y \backslash n, z \backslash n) \restriction n & \text{if defined} \\ * \text{ else.} \end{cases}$$

We let $s_{\beta,\gamma,\delta}(N, M) \restriction l = *$ when $l \geq n$. Finally we are able to define what it means for a triple of ordinals (β, γ, δ) to code a real r.

Definition 13. *For a triple of limit ordinals (β, γ, δ), we say that it codes a real $r \in 2^\omega$ if there is a continuous increasing sequence $(N_\xi \,:\, \xi < \omega_1)$ of countable sets of ordinals , also called a reflecting sequence, whose union is δ and which satisfies that whenever $\xi < \omega_1$ is a limit ordinal then there is a $\nu < \xi$ such that*

$$r = \bigcup_{\nu < \eta < \xi} s_{\beta,\gamma,\delta}(N_\eta, N_\xi).$$

Witnesses to the coding can be added with a proper forcing. Moreover there is a certain amount of control for fixed triples of ordinals and the behavior of continuous, increasing sequences on them:

Theorem 14 (Caicedo-Velickovic). (†) *Given ordinals $\omega_1 < \beta < \gamma < \delta < \omega_2$ of cofinality ω_1, there exists a proper notion of forcing $\mathbb{P}_{\beta\gamma\delta}$ such that after forcing with it the following holds: There is a reflecting, i.e., increasing and continuous sequence $(N_\xi \,:\, \xi < \omega_1)$ such that $N_\xi \in [\delta]^\omega$ whose union is δ such that for every limit $\xi < \omega_1$ and every $n \in \omega$ there is $\nu < \xi$ and $s^n_\xi \in 2^n$ such that*

$$s_{\beta\gamma\delta}(N_\eta, N_\xi) \restriction n = s^n_\xi$$

holds for every η in the interval (ν, ξ). We say then that the triple (β, γ, δ) is stabilized.

(‡) *Further if we fix a real r there is a proper notion of forcing $\mathbb{P}_r$ such that the forcing will produce for a triple of ordinals $(\beta_r, \gamma_r, \delta_r)$ of size and cofinality $\aleph_1$ a reflecting sequence $(P_\xi : \xi < \omega_1)$, $P_\xi \in [\delta_r]^\omega$ such that $\bigcup_{\xi<\omega_1} P_\xi = \delta_r$ and such that for every limit $\xi < \omega_1$ there is a $\nu < \xi$ such that*

$$\bigcup_{\nu<\eta<\xi} s_{\beta_r\gamma_r\delta_r}(P_\eta, P_\xi) = r.$$

We say then that the real r is determined by the triple $(\beta_r, \gamma_r, \delta_r)$.

Note here that for (‡) there is no way of controlling the triple of ordinals (β, γ, δ) for which $\mathbb{P}_r$ adds an increasing sequence $(P_\xi : \xi < \aleph_1)$ of countable sets of ordinals which code r.

The coding can be used to generically produce a hierarchy on $H(\omega_2)$ which is robust under stationary set preserving notions of forcing. Recall that two reals r, s are almost disjoint if $r \cap s$ is finite. Using our fixed ladder system $\vec{C}$ we can define from $\vec{C}$ an almost disjoint family of reals $F_{\vec{C}} := (r_\alpha : \alpha < \omega_1)$. Then, if $X \subset \omega_1$ is arbitrary, the almost disjoint coding forcing introduces a new real r_X such that the following holds:

$$\forall \xi < \omega_1 \, (\xi \in X \text{ iff } r_X \cap r_\xi \text{ is finite}).$$

It is well known that this forcing is ccc, therefore proper.

Definition 15. *Fix a ladder system $\vec{C}$ and let $F_{\vec{C}}$ be a family of almost disjoint reals which is definable from $\vec{C}$. Let $\mathsf{T}_{\vec{C}}$ denote the following list of axioms:*

1. $\forall x(|x| \leq \aleph_1)$,
2. ZF^-,
3. *Every subset of ω_1 is coded by a real, relative to the almost disjoint family $F_{\vec{C}}$.*
4. *Every triple of limit ordinals is stabilized in the sense of † using $\vec{C}$.*
5. *Every real is determined by a triple of ordinals in the sense of ‡ using $\vec{C}$.*

A highly useful feature of models of $\mathsf{T}_{\vec{C}}$ is that they are uniquely determined by their height, consequentially the uncountable $\mathsf{T}_{\vec{C}}$-models form a hierarchy below $H(\omega_2)$.

Theorem 16. *Let $\vec{C}$ be a ladder system in M, assumed to be a transitive model of $T_{\vec{C}}$. Then, M is the unique model of $\mathsf{T}_{\vec{C}}$ of height $M \cap Ord$.*

Proof. Assume that M and M' are transitive, $M \cap Ord = M' \cap Ord$, $\vec{C} \in M \cap M'$, which implies that M and M' have the right ω_1, and both M and M' are $\mathsf{T}_{\vec{C}}$ -models. We work towards a contradiction, so assume that $X \in M$ yet $X \notin M'$. As every set in M has size at most $\aleph_1$ we can assume that $X \subset \omega_1$, hence there is a real $r_X \in M$ which codes X with the help of the the almost disjoint family $F_{\vec{C}}$. Now r_X is itself coded by a triple of ordinals $(\beta, \gamma, \delta) \in M$, thus there is a reflecting sequence $(N_\xi : \xi < \omega_1) \in M$ witnessing that r_X is determined by (β, γ, δ). As $M \cap Ord = M' \cap Ord$, (β, γ, δ) is in M' as well, and there is a reflecting sequence $(P_\xi : \xi < \omega_1) \in M'$ which witnesses that (β, γ, δ) is stabilized in M'. The set $C := \{\xi < \omega_1 : P_\xi = N_\xi\}$ is a club on ω_1, hence if η is a limit point of C, the reflecting sequence $(P_\xi : \xi < \omega_1) \in M'$ will stabilize at η and compute r_X, hence X is an element of M' which is a contradiction. $\square$

2 NS_{ω_1} saturated and a projective wellorder.

The goal of this section is the proof of the following result:

Theorem 17. *Assume that M_1 exists, then there exists a generic extension $M_1[G]$ of M_1 such that in $M_1[G]$ NS_{ω_1} is $\aleph_2$-saturated and there is a lightface Σ^1_4-definable wellorder of the reals.*

Its proof is organized as follows. We start with M_1 as our ground model, let Λ be its Woodin cardinal. We will use an RCS-iteration of length Λ guided by a $\diamondsuit$-sequence $(a_\alpha)_{\alpha < \Lambda}$, which will seal off long maximal antichains in $P(\omega_1)/\mathrm{NS}_{\omega_1}$ as long as the forcing is semiproper, code reals into triples of ordinals, stabilize sets of triples of ordinals, add almost disjoint reals, and constantly localize the information we obtained during the process into subsets of ω_1 whose information can be read off already by suitable countable transitive models of ZF^-. As the factors are all semiproper, the iteration will be a semiproper, hence stationary-preserving forcing. In the end the Woodin cardinal Λ will be used to show that NS_{ω_1} in fact is $\aleph_2$-saturated in the final model. Yet we will have produced a sequence of $\mathsf{T}_{\vec{C}}$-models whose heights are unbounded in ω_2, and the fact that we did produce local witnesses for being a $\mathsf{T}_{\vec{C}}$-model will guarantee us that the wellorder can be seen in suitable, countable, transitive models which ultimately yield a Σ^1_4-definable wellorder.

2.1 Coding the reals

We will use the $\mathsf{T}_{\vec{C}}$ models to set up a wellorder of the reals. It is a fact that every transitive $\mathsf{T}_{\vec{C}}$ model M can define a wellorder $<_M$ of $(\omega^\omega)^M$ via letting $r <_M s$ if and only if the antilexicographically least triple of ordinals $(\alpha_r, \beta_r, \gamma_r)$ which codes r in the sense of (‡) is less than the antilexicographically least triple which codes s. If we assume that V is a universe such that

$H(\omega_2) \models \mathsf{T}_{\vec{C}}$, then the local wellorders $<_M$ of the $\mathsf{T}_{\vec{C}}$-models $M \in V$ can be put together in a straightforward way to form a new wellorder of ω^ω.

Definition 18. *Assume that V is a universe such that $H(\omega_2) \models \mathsf{T}_{\vec{C}}$. We define a function $f : \omega^\omega \to Ord$; for a real r we let $f(r)$ be the least ordinal η such that r is in the unique $\mathsf{T}_{\vec{C}}$-model of height η. Then for $r, s \in \omega^\omega$ we set $r < s$ if and only if $f(r) < f(s)$ or $f(r) = f(s) = \alpha$ and $r <_{M_\alpha} s$, for M_α being the unique $\mathsf{T}_{\vec{C}}$-model of height α, and $<_{M_\alpha}$ being the local wellorder defined above.*

The just defined wellorder is very robust.

Lemma 19. *Assume that V is some universe such that $H(\omega_2) \models \mathsf{T}_{\vec{C}}$. The order $<$ has a $\Delta_1(\vec{C})$ definition. Consequentially any transitive ZF^--model M which contains $\vec{C}$ and satisfies that every real is contained in some $\mathsf{T}_{\vec{C}}$-model will correctly compute the relation $x < y$ for $x, y \in M$, i.e., $M \models x < y$ if and only if $V \models x < y$.*

Proof. The function f which maps every real to the height of the least $\mathsf{T}_{\vec{C}}$-model containing it is $\Delta_1(\vec{C})$. The definition of the local wellorder $<_{M_\alpha}$ is $\Delta_1(\vec{C}, \alpha)$ so $<$ is defined via a $\Delta_1(\vec{C})$-formula. $\square$

2.2 Definition of the iteration

Next, we describe how to code reals nicely while making NS_{ω_1} $\aleph_2$-saturated. In order to get NS_{ω_1} $\aleph_2$-saturated, we need an RCS-iteration of length Λ, where Λ is the Woodin cardinal. We fix a $\diamondsuit$-sequence $(a_\alpha : \alpha < \Lambda)$ in the ground model $V = M_1$ and use the already introduced, cofinal set of M_1-initial segments $\mathcal{I}$ whose set of codes is Π^1_2-definable, to construct a ladder system $\vec{C}$ which is Σ^1_3-definable in the codes and whose definition will continue to produce a ladder system in ω_1-preserving outer models of M_1. Simply let $(\alpha, C_\alpha) \in \vec{C}$ if and only if there is a countable M_1 initial segment $M \in \mathcal{I}$ which contains (α, C_α) and which sees that C_α is the $<_M$-least set in M (where $<_M$ denotes the usual definable wellorder on the premouse M) which is cofinal in α and has order type ω. The definition is Σ^1_3. This particular $\vec{C}$ will be our fixed ladder system we use in our proof. We additionally fix an almost disjoint family of reals $F_{\vec{C}}$ of size $\aleph_1$ which we can compute from the ladder system $\vec{C}$, via turning the set of reals which code elements of $\vec{C}$ into an almost disjoint family of reals using the standard trick of turning an arbitrary set of reals into an almost disjoint family. As an alternative, we could also use again the wellorder $<_M$ to define an almost disjoint family F, both ways work but we stick with the first.

We describe first informally how the iteration looks. As always we have stages which are used to code information yielding the definable wellorder and stages where we seal off long antichains in $P(\omega_1)/\mathrm{NS}_{\omega_1}$. We ensure that we code all the reals we generate during the iteration into triples of ordinals (β, γ, δ) using the proper forcing of (‡). At the same time, we ensure that all the triples of ordinals below ω_2 stabilize using the forcing described in (†), and that every set $X \subset \omega_1$ we create will be coded by a real r_X relative to the almost disjoint family of reals $F_{\vec{C}}$. Additionally, whenever our $\Diamond$-sequence hits the name of a long antichain in $P(\omega_1)/\mathrm{NS}_{\omega_1}$ we seal it off, if doing so is semiproper. As we have stationarily many inaccessible cardinals below the Woodin Λ, we will stationarily often hit inaccessible stages α such that the model $(M_1)_\alpha[G_\alpha]$ (where we write $(M_1)_\alpha$ for $\mathcal{J}_\alpha^{M_1}$) is equal to $H(\omega_2)^{M_1[G_\alpha]}$ and satisfies the already defined theory $\mathsf{T}_{\vec{C}}$. So

(1) $(M_1)_\alpha[G_\alpha] \models \mathsf{ZF}^-$ and $\forall x(|x| \leq \aleph_1)$.

(2) $(M_1)_\alpha[G_\alpha] \models \forall \beta, \gamma, \delta((\beta, \gamma, \delta)$ is stabilized),

(3) $(M_1)_\alpha[G_\alpha] \models \forall r \in \omega^\omega \exists (\beta_r, \gamma_r, \delta_r)$ (r is determined by $(\beta_r, \gamma_r, \delta_r)$).

(4) $(M_1)_\alpha[G_\alpha] \models \forall X(X \subset \omega_1 \, \exists r_X \in \omega^\omega(r_X$ codes X with the help of the almost disjoint family $F_{\vec{C}}$)).

Whenever we hit such a stage everything $(M_1)_\alpha[G_\alpha]$ sees about $<$ will be preserved in all future extensions in our iteration by Lemma 19. Thus, we will additionally localize the $\mathsf{T}_{\vec{C}}$-model $(M_1)_\alpha[G_\alpha]$ i.e., we add a subset Y_α of ω_1 such that every countable transitive model N of ZF^- which contains $Y_\alpha \cap \omega_1^N$ will also contain $\vec{C} \restriction \omega_1^N$ and see that there is a $\mathsf{T}_{\vec{C} \restriction \omega_1^N}$-model which witnesses true assertions about the wellorder $<$. This uses a proper forcing again. As all the iterands are proper or semiproper, using an RCS-iteration will yield a semiproper notion of forcing. In the end, we will argue that indeed NS_{ω_1} is saturated and there is a Σ^1_4-definable wellorder of the reals.

We start now with a more detailed description of how the iteration should look. We will construct the iteration recursively, so assume that $\alpha < \Lambda$ and we have already constructed $\mathbb{P}_\beta$ for $\beta \leq \alpha$. We define the forcing $\dot{\mathbb{Q}}_\alpha$ in $V^{\mathbb{P}_\alpha}$ as follows:

(i) Assume that a_α is a $\mathbb{P}_\alpha$-name of a real r_α. Then we let $\dot{\mathbb{Q}}_\alpha$ be the $\mathbb{P}_\alpha$-name of the forcing which codes r_α into a triple of ordinals $(\beta_{r_\alpha}, \gamma_{r_\alpha}, \delta_{r_\alpha})$, such that $\beta_{r_\alpha}, \gamma_{r_\alpha}, \delta_{r_\alpha} < \omega_2$ and using the already fixed $\vec{C}$-sequence. This forcing is followed by considering all the triples of ordinals $(\beta', \gamma', \delta')$ which are antilexicographically below $(\beta_{r_\alpha}, \gamma_{r_\alpha}, \delta_{r_\alpha})$ and which have not been stabilized yet. We use a countable support iteration of forcings which stabilize each such triple $(\beta', \gamma', \delta')$. As a summary $\dot{\mathbb{Q}}_\alpha$ is an ω_1-long iteration of proper forcings with countable support, resulting in a proper forcing, and we obtain a model where the real r_α is coded into the triple $(\beta_{r_\alpha}, \gamma_{r_\alpha}, \delta_{r_\alpha})$ with the help of the ladder system $\vec{C}$, and each other triple of ordinals below it will be stabilized.

(ii) Assume that α is an inaccessible, further that a_α is the $\mathbb{P}_\alpha$-name of a maximal antichain S_α in $P(\omega_1)/\mathrm{NS}_{\omega_1}$, and assume that the sealing forcing $\mathbb{S}(S_\alpha)$ is semiproper. Then force with it, i.e., let $\dot{\mathbb{Q}}_\alpha$ be $\mathbb{S}(S_\alpha)$.

Otherwise force with $Col(2^{\aleph_2}, \aleph_1)$, the usual Lévy collapse which collapses $2^{\aleph_2}$ down to $\aleph_1$.

(iii) If a_α is a $\mathbb{P}_\alpha$-name of a subset X of ω_1 then use almost disjoint coding forcing to add a real r_X which codes X with the help of the almost disjoint family of reals $F_{\vec{C}}$.

(iv) If α is an inaccessible cardinal and if $(M_1)_\alpha[G_\alpha]$ is such that $(M_1)_\alpha[G_\alpha]$ equals $H(\omega_2)^{M_1[G_\alpha]}$ and $(M_1)_\alpha[G_\alpha] \models \mathsf{T}_{\vec{C}}$, then we first collapse its size down to $\aleph_1$, and subsequently add a subset Y of ω_1 which should code the $\mathsf{T}_{\vec{C}}$-model $(M_1)_\alpha[G_\alpha]$ in a more suitable way. This set Y will then be coded into a real r_Y using almost disjoint coding forcing.

The points (i), (ii), and (iii) are clear, thus we shall discuss (iv) in detail: So assume that α is an inaccessible cardinal, G_α is the generic filter for the iteration we have produced so far, $\vec{C}$ is the ladder system whose codes form a Σ^1_3-definable subset of the reals, $F_{\vec{C}}$ the almost disjoint family of reals we define from $\vec{C}$, $(M_1)_\alpha[G_\alpha] = H(\omega_2)^{M_1[G_\alpha]}$, and $(M_1)_\alpha[G_\alpha] \models \mathsf{T}_{\vec{C}}$.

We collapse the size of $(M_1)_\alpha[G_\alpha]$ to $\aleph_1$ using Lévy-collapse and let H be the generic filter over $M_1[G_\alpha]$. For the following fix a pair of Δ_1- definable functions dec_1 and dec_2 (*dec* for decoding) which act on subsets of ω_1.

Fact 20. *In $M_1[G_\alpha][H]$ there is a set $X_\alpha \subset \omega_1$ such that*

1. *$dec_1(X_\alpha) = \vec{C}$, and for every limit ordinal $\xi < \omega_1$, $dec_1(X_\alpha \cap \xi) = \vec{C} \restriction \xi$.*
2. *$dec_2(X_\alpha) = (M_1)_\alpha[G_\alpha]$.*

The construction of such a set X_α is straightforward. As a consequence, every transitive model M of ZF^- which contains X_α will see that $dec_1(X_\alpha)$ is a ladder system and $dec_2(X_\alpha)$ is the unique $\mathsf{T}_{dec_1(X_\alpha)}$-model of height α.

The goal now is to rewrite X_α into a set $Y_\alpha \subset \omega_1$ such that not only $\aleph_1$-sized, but already suitable, countable ZF^--models M which contain $Y_\alpha \cap \omega_1^M$ see that $dec_1(Y_\alpha \cap \omega_1^M)$ is a ladder system and $dec_2(Y_\alpha \cap \omega_1^M)$ is a $\mathsf{T}_{dec_1(Y_\alpha \cap \omega_1^M)}$-model. We can force the existence of such a set Y_α with a proper notion of forcing. In the next Lemma we will use our suitable decoding functions dec_i from above, but we demand that $dec_i(Y)$ will act only on the *even* elements of Y. To be more precise for every set of ordinals Y we collect the even elements Y_{even} of Y and whenever we write $dec_i(Y)$ we actually mean $dec_i(Y_{even})$. This facilitates the notation slightly.

Lemma 21. *Let the set X_α be just as above. There is a proper notion of forcing $\mathbb{R}$ which introduces a set $Y_\alpha \subset \omega_1$ such that if H' is an $\mathbb{R}$-generic filter, $M_1[G_\alpha][H][H']$ satisfies that for any countable, transitive N, $\xi := \omega_1^N$, which contains $Y_\alpha \cap \xi$ then for our fixed, recursively definable decoding functions dec_i, which act only on the even entries of Y_α, the following holds in N:*

1. *$dec_1(Y_\alpha \cap \xi) = \vec{C} \restriction \xi$.*
2. *$dec_2(Y_\alpha \cap \xi)$ is a $\mathsf{T}_{dec_1(Y_\alpha \cap \xi)}$-model.*

Proof. Working in $M_1[G_\alpha][H]$ we have that $(M_1)_\alpha[G_\alpha]$ is a model $\mathsf{T}_{\vec{C}}$ of size $\aleph_1$. Fix a model of the form $(M_1)_\eta[G_\alpha][H]$ for $\eta > \alpha$ which contains the set $X_\alpha \subset \omega_1$ from above and consider the club C of countable, elementary submodels of $(M_1)_\eta[G_\alpha][H]$ which contain X_α. If we pick an arbitrary $M \in C$ then M will contain $\vec{C}$ and $(M_1)_\alpha[G_\alpha]$, thus for the transitive collapse $\bar{M}$ of M we have that

$$\bar{M} \models dec_1(X_\alpha \cap \omega_1^{\bar{M}}) \text{ is the ladder system } \vec{C} \restriction \omega_1^{\bar{M}} \text{ for } \omega_1^{\bar{M}}.$$

$$\bar{M} \models dec_2(X_\alpha \cap \omega_1^{\bar{M}}) \text{ is a } \mathsf{T}_{dec_1(X_\alpha \cap \omega_1^{\bar{M}})}\text{-model.}$$

In order to get the full statement of the Lemma, we add additional information to X_α which yields Y_α such that any countable transitive model N of ZF^-, which contains $Y_\alpha \cap \omega_1^N$ must have its ω_1 to be an ω_1^M for some $M \in C$. To achieve this, we use forcing.

Let $\mathbb{R}$ be the following partial order: conditions $p \in \mathbb{R}$ are ω_1-Cohen conditions, i.e., functions from limit ordinals $\xi < \omega_1$ with $\omega^\xi = \xi$ (in terms of ordinal exponentiation) to 2, ordered by end-extension which additionally satisfy:

1. the even ordinals of $\{\eta < \xi : p(\eta) = 1\}$, where $\xi = dom(p)$ code the set $X_\alpha \cap \xi$.
2. for every limit ordinal $\zeta \leq dom(p)$ with $\omega^\zeta = \zeta$, $p \restriction \zeta$ satisfies that whenever $M \models \mathsf{ZF}^-$ is countable and transitive and $\zeta = \omega_1^M$ and $(p \restriction \zeta) \in M$ then
   (a) if we consider $p \restriction \zeta$ as a subset of ζ, $M \models dec_1(p \restriction \zeta) = \vec{C} \restriction \zeta$.
   (b) $M \models dec_2(p \restriction \zeta)$ is a $\mathsf{T}_{\vec{C} \restriction \zeta}$-model.

Note that whenever we do have a condition $p \in \mathbb{R}$, and $\xi < \omega_1$ is a limit ordinal, we can extend p to a condition $q < p$ such that $\xi \in dom(q)$. This is clear as we can pick a function q end-extending p with domain some countable limit ordinal $\zeta > \xi$ and write into the odd ordinals of the first ω-block of q following $dom(p)$ a surjection of ζ to ω, while the even entries of

q in the interval $(dom(p), \zeta)$ just code $X_\alpha \cap \zeta$. Then no countable transitive model M of ZF^-, which contains q can have its ω_1 in the interval $(dom(p), \zeta]$, thus the second property for being a condition in $\mathbb{R}$ is satisfied automatically.

Consequentially, the set $D_\eta := \{p \in \mathbb{R} : \eta \in dom(p)\}$ is dense for every $\eta < \omega_1$ and the generic will produce a subset of ω_1, Y_α with the desired properties for countable, transitive models of ZF^- in $M_1[G_\alpha][H]$. This already suffices as we will see below that the forcing $\mathbb{R}$ is also ω-distributive.

What is left is to show that the forcing $\mathbb{R}$ is proper: for that we pick the $(M_1)_\eta[G_\alpha][H]$ from above and recall that the club C was defined to be the set of all countable elementary submodels of $(M_1)_\eta[G_\alpha][H]$ which contain the set X_α. If $M \in C$, and $p \in \mathbb{R} \cap M$ then we shall construct a $q < p$ which is $(M, \mathbb{R})$-generic. We list all the dense sets D_n in M and recursively construct a descending sequence of conditions starting at $p = p_0 > p_1 > \dots$ such that $p_n \in D_n$ and such that the supremum of the $(dom(p_n))$'s equals $\omega_1 \cap M$. If we can show that the limit p_ω is a condition in $\mathbb{R}$, we are done.

Thus we shall argue that whenever $\xi \leq dom(p_\omega)$, $p_\omega \cap \xi$ is contained in a countable, transitive model $N \models \mathsf{ZF}^-$ such that $\xi = \omega_1^N$ then it will satisfy that $dec_1(p_\omega \restriction \xi)$ equals $\vec{C} \restriction \xi$ and $dec_2(p_\omega \restriction \xi)$ is a $\mathsf{T}_{\vec{C} \restriction \xi}$-model. This is clear by definition of $\mathbb{R}$ for every $\xi < dom(p_\omega)$. If $\xi = dom(p_\omega)$ then as $\xi = \omega_1^{\bar{M}}$, $M \in C$ we know by the above that

$$\bar{M} \models dec_1(X_\alpha \cap \xi) \text{ is the ladder system } \vec{C} \restriction \xi \text{ and}$$

$$\bar{M} \models dec_2(X_\alpha \cap \xi) \text{ is a } \mathsf{T}_{dec_1(X_\alpha \cap \xi)}\text{-model.}$$

Consequentially if $N \models \mathsf{ZF}^-$ is a countable, transitive model which contains p_ω and $dom(p_\omega) = \xi = \omega_1^N$, then N will also contain $X_\alpha \cap \xi$, as this is coded into the even entries of p_ω. As the decoding functions dec_i are absolute for transitive models, N will compute the information written into $X_\alpha \cap \xi$ just in the same way as $\bar{M}$ does. The notion of being a ladder system and the notion of being a T-model is absolute for transitive models as well, thus N will satisfy that $dec_1(p_\omega \restriction \xi)$ equals $\vec{C} \restriction \xi$ and $dec_2(p_\omega \restriction \xi)$ is a $\mathsf{T}_{\vec{C} \restriction \xi}$-model. So p_ω is indeed a condition in $\mathbb{R}$ and the forcing is proper. Note that the same argument shows that $\mathbb{R}$ is also ω-distributive. $\square$

It is important to note the following: assume that $dec_2(X_\alpha) = H(\omega_2)^{M_1[G_\alpha]}$ thinks that for two reals x and y, $x < y$ holds. Then, any countable transitive model N, $\xi = \omega_1^N$ which contains $Y_\alpha \cap \xi, x$ and y will see that $dec_2(Y_\alpha \cap \xi) \models x < y$. This is immediate from the above proof and will play an important role later.

In the next step, we will code the set $Y_\alpha \subset \omega_1$ into a real r_{Y_α}, using almost disjoint coding relative to our fixed almost disjoint family of reals $F_{\vec{C}} = (r_\xi : \xi < \omega_1)$. Thus, we introduce a real r_{Y_α} such that the following holds:

$$\forall \xi < \omega_1 \, (\xi \in Y_\alpha \text{ iff } r_{Y_\alpha} \cap r_\xi \text{ is finite}).$$

It is well known that this forcing is ccc, therefore proper. This ends the definition and the discussion of the forcing we use in Case (iv) in the definition of our iteration. We close this section with a discussion of what we have produced following the definition of the iteration.

The effect of the real r_{Y_α} is that any countable, transitive model M of some reasonable fragment of ZFC which contains it will also contain the set Y_α as long as M knows enough about the almost disjoint family $F_{\vec{C}}$. These models will play a important role in our proof thus we define rigorously what we mean with a suitable model.

Definition 22. *A countable, transitive model M of ZF^- is said to be suitable if $(M_1)_{\omega_1^M} \in M$ and every $\alpha < \omega_1^M$ is already countable in $(M_1)_{\omega_1^M}$.*

Note that the statement "N is a suitable model" is Σ^1_3 for the N's whose ω_1 coincides with the ω_1 of an $(M_1)_\eta \in \mathcal{I}$, where $\mathcal{I}$ is the Π^1_2-definable family of countable initial segments of M_1, as we can write "N is suitable" if and only if $\exists M(M \in \mathcal{I} \wedge M \in N \wedge \omega_1^N = \omega_1^M)$. It can also be written in a Π^1_3-way, as $\forall M(M \in \mathcal{I} \wedge \omega_1^M = \omega_1^N \rightarrow M \in N)$ yields that N is suitable. If we want to quantify over all suitable models N, we can use the latter formulation as well: a formula $\forall N(N \text{ is suitable} \rightarrow \varphi(N, \ldots))$ can be equivalently written as $\forall N \forall M(M \in \mathcal{I} \wedge \omega_1^N = \omega_1^M \wedge M \in N \rightarrow \varphi(N, \ldots))$ which has the advantage that its antecedent is a Π^1_2-formula.

As already mentioned above, suitable models containing r_{Y_α} also contain Y_α up to their local ω_1. Indeed, if M is suitable and $r_{Y_\alpha} \in M$, then M will contain $\vec{C} \cap \omega_1^M$, as it can use $(M_1)_{\omega_1^M}$ and take advantage of the fact that $\vec{C}$ is uniformly definable in all initial segments of M_1. Thus $F_{\vec{C}} \cap \omega_1^M \in M$, as the latter is definable from the ladder system in an absolute way. So M can decode from r_{Y_α} and obtain $Y_\alpha \cap \omega_1^M$. We have already shown that the containment of Y_α causes every countable transitive model to see that it also contains a local $\mathsf{T}_{\vec{C}}$ model. To summarize the above, in Case (iv) of the definition of the iteration we force with a three step iteration of proper forcings which introduce a real r_{Y_α} such that the following holds:

♡ Every suitable model M of ZF^- which contains r_{Y_α} thinks that r_{Y_α} codes $Y_\alpha \cap \omega_1^M$ such that $dec_1(Y_\alpha \cap \xi) = \vec{C} \restriction \xi$. Further, $dec_2(Y_\alpha \cap \xi)$ is a $\mathsf{T}_{dec_1(Y_\alpha \cap \xi)}$-model. Moreover if $x \in M$ and $y \in M$ are two reals such that $M_1[G_\alpha] \models x < y$, then $dec_2(Y_\alpha \cap \xi) \models x < y$.

2.3 NS_{ω_1} is saturated and a projective wellorder.

Let G be a generic filter for the Λ-long iteration we defined in the last section. We shall discuss the important properties of our resulting universe $M_1[G]$ and eventually show that the model will indeed satisfy that NS_{ω_1} is saturated and there is a Σ^1_4 definable wellorder of the reals. Note first that $M_1[G]$ will contain many $\mathsf{T}_{\vec{C}}$-models, in fact $H(\omega_2)^{M_1[G]}$ is a $\mathsf{T}_{\vec{C}}$-model itself and $\{\alpha < \omega_2 : \exists N_\alpha (N_\alpha \text{ is the } \mathsf{T}_{\vec{C}} \text{ model of height } \alpha)\}$ is unbounded (in fact stationary) in ω_2. We will use the $\mathsf{T}_{\vec{C}}$-models to witness the wellorder $<$ of the reals. Recall that $x < y$ was defined to hold whenever the least $\mathsf{T}_{\vec{C}}$-model containing x has shorter ordinals height than the least such model for y or else if $M \models x <_M y$. It is a direct consequence of Lemma 19 that $\mathsf{T}_{\vec{C}}$-models can be used to witness the relation $x < y$, i.e., for arbitrary reals in $M_1[G]$ it holds that $x < y$ if and only if there is a $\mathsf{T}_{\vec{C}}$-model M which itself contains unboundedly many $\mathsf{T}_{\vec{C}}$-models such that $M \models x < y$. Thus, in order to obtain a projective wellorder of the reals it is sufficient to find a projective way of defining $\mathsf{T}_{\vec{C}}$-models. In $M_1[G]$ this can be done by the way we defined our iteration.

Lemma 23. *There is a Π^1_3-formula $\theta(x)$ for which the following holds in $M_1[G]$:*

> $\forall r \in \omega^\omega$*(if $\theta(r)$ holds then r is the almost disjoint code for a $Y \subset \omega_1$ and $dec_2(Y)$ is a $\mathsf{T}_{\vec{C}}$-model relative to the almost disjoint family $F_{\vec{C}}$).*

Proof. First we let $\psi(x)$ be the Σ^1_3-formula which implies that x is a code for a suitable model, i.e., $\psi(x) := \exists z\, (z \text{ is a code for an element of } \mathcal{I} \wedge \omega_1^z = \omega_1^x \wedge z \in x) \wedge x \models \mathsf{ZF}^-$, where $\mathcal{I}$ is the Π^1_2-definable family of countable initial segments of M_1. Recall that if M is a suitable model, we can use $(M_1)_{\omega_1^M} \in M$ to define $\vec{C} \restriction \omega_1^M$, by picking always the $<_{(M_1)_{\omega_1^M}}$-least real coding a cofinal set of ordertype ω. Once we have $\vec{C} \restriction \omega_1^M$, we also get the almost disjoint family $F_{\vec{C}} \restriction \omega_1^M$. Let $\sigma(x, y)$ be the formula, stating that x is an M_1-initial segment and y being the ladder system we get, when forming a ladder system recursively via always picking the $<_x$-least real. Note that $\sigma(x, y)$ can be written as a Π^1_2-formula: $\sigma(x, y) \leftrightarrow (x \in \mathcal{I}$ and $x \models y$ is the ladder system one obtains when always picking the $<_x$-least real). Likewise, there is a Π^1_2-formula $\varpi(x, y)$ which implies that x is an M_1-initial segment and y is the almost disjoint family $F_{\vec{C}} \restriction \omega_1^x$.

We let $\theta(r)$ be the following formula:

$\theta(r) \leftrightarrow \forall M(\psi(M)$, and $r \in M$ and $\exists F(\varpi((M_1)_{\omega_1^M}, F)$ and $M \models$ "r is the almost disjoint code for a set $Y \subset \omega_1$ using F" and $\sigma((M_1)_{\omega_1^M}, dec_1(Y))$ then $dec_2(Y)$ is a $\mathsf{T}_{dec_1(Y)}$-model).

Note that $\theta(r)$ is of the form $\forall M(\Sigma^1_3 \wedge \Delta_0 \wedge \exists F(\Pi^1_2 \wedge \Delta^1_2) \wedge \Pi^1_2 \rightarrow \Delta^1_2)$, thus $\theta(r)$ is a Π^1_3-formula. In plain words $\theta(r)$ says that every suitable model M containing r will decode out of r a subset Y of ω_1^M which in turn codes two things namely a ladder system in M (which also is the ladder system M would construct with the help of its M_1 initial segment) and a T-model in M relative to that ladder system. Note further that we ensured that in $M_1[G]$ there are plenty of such reals r for which $\theta(r)$ holds, as we cofinally often added them whenever we were in Case (iv) in the definition of our iteration. This is a consequence of the fact that $\heartsuit$ holds in that situation for such r.

What is left is to show that whenever $\theta(r)$ holds in $M_1[G]$, then r is indeed an almost disjoint code relative to the almost disjoint family $F_{\vec{C}}$ for a $\mathsf{T}_{\vec{C}}$ model. Assume not, thus r is the almost disjoint code of a set which is not a $\mathsf{T}_{\vec{C}}$-model even though $\theta(r)$ holds. We pick a large enough, M_1-inaccessible cardinal κ and some suitable $\eta < \Lambda$ such that $(M_1)_\kappa[G_\eta]$ thinks that the real r does code Y and $dec_2(Y)$ is not a $\mathsf{T}_{\vec{C}}$-model. So $((M_1)_\kappa[G_\eta], \in, (M_1)_{\omega_1})$ satisfies that $\vec{C}$ is the outcome when applying the Σ^1_3-definition of our ladder system in $(M_1)_{\omega_1}$, it satisfies that $F_{\vec{C}}$ is the $(M_1)_{\omega_1}$-evaluated almost disjoint family and $Y \subset \omega_1$ is the decoded r, yet $dec_2(Y)$ is not a $\mathsf{T}_{\vec{C}}$-model.

We can always choose a countable elementary submodel $(N, \in, (M_1)_{\omega_1}) \prec ((M_1)_\kappa[G_\eta], \in, (M_1)_{\omega_1})$ containing r such that its transitive collapse $(\bar{N}, \in, (M_1)_{\omega_1^{\bar{N}}})$ is such that $(M_1)_{\omega_1^{\bar{M}}} \in \mathcal{I}$, thus $\psi(\bar{N})$ holds. Moreover, $\bar{N}$ models the rest of the antecedent of $\theta(r)$ yet still thinks that r does not code a $\mathsf{T}_{\vec{C} \upharpoonright \omega_1^{\bar{N}}}$-model by elementarity of N . But then $\bar{N}$ witnesses that $\theta(r)$ is not true which is a contradiction. $\square$

We will use the projective formula for being a $\mathsf{T}_{\vec{C}}$-model to find witnessing models which are correct about the wellorder $<$ of the reals. As $\mathsf{T}_{\vec{C}}$-models are correct about $<$ we can internalize the wellorder, thus arriving at a Σ^1_4-definition.

Lemma 24. *There is a* Σ^1_4*-formula* $\Phi(x, y)$ *such that in* $M_1[G]$*,* $x < y$ *is true if and only if* $\Phi(x, y)$ *holds.*

Proof. We take advantage of the fact that $x < y$ if and only if there is a $\mathsf{T}_{\vec{C}}$-model N which satisfies that $x < y$ holds. Recall the formula $\theta(r)$ which asserts that every suitable model M will decode out of r a ladder system and a T-model relative to it. Now all that is left is to add that this local T-model in fact witnesses $x < y$.

Let $\Phi(x, y)$ be the formula

> $\exists r \forall M($if $\psi(M) \wedge r, x, y \in M \wedge \exists F \in M(\varpi((M_1)_{\omega_1^M}, F)$ and $M \models \backslash r$ and F code a set $Y \subset \omega_1$" and $\sigma((M_1)_{\omega_1^M}, dec_1(Y))$ then $dec_2(Y) \in M$ is a $\mathsf{T}_{dec_1(Y)}$-model which sees $x < y))$.

$\Phi(x, y)$ is of the form $\exists r \forall M(\Sigma^1_3 \wedge \Delta_0 \wedge \exists F(\Pi^1_2 \wedge \Delta^1_2) \wedge \Pi^1_2 \rightarrow \Delta^1_2))$, hence Σ^1_4. We shall show that in $M_1[G]$, $x < y$ is true if and only if $\Phi(x, y)$ is true.

For the direction from left to right, note that if $x < y$ then there will be a sufficiently large $\alpha < \Lambda$ such that $(M_1)_\alpha[G_\alpha]$ is a $\mathsf{T}_{\vec{C}}$-model and the $H(\omega_2)$ of $M_1[G_\alpha]$. We can assume that $\{x, y\} \in M_1[G_\alpha]$. We know already that at such a stage we will add a real r such that $\varphi(r)$ and $\heartsuit$ holds, thus $\Phi(x, y)$ is true.

For the direction from right to left note that when $\Phi(x, y)$ holds, this means in particular that $\theta(r)$ holds. By the last Lemma, r is the almost disjoint code for a $\mathsf{T}_{\vec{C}}$-model, and by the last paragraph it sees $x < y$. Thus $x < y$ is true in $M_1[G]$. $\square$

What is left is to show that in $M_1[G]$ the nonstationary ideal NS_{ω_1} is indeed $\aleph_2$-saturated. But this does not cause any problems as the coding forcings were all seen to be proper, the sealing forcings were only used when semiproper and we used RCS-iteration for the limit steps. Therefore the iteration yields a semiproper, thus stationary set preserving extension of M_1 and we can just repeat Shelah's proof that NS_{ω_1} is $\aleph_2$-saturated in the final model.

Theorem 25. *If G denotes the generic filter for the iteration then in $M_1[G]$ the nonstationary ideal NS_{ω_1} is $\aleph_2$-saturated.*

Proof. The proof draws heavily from R. Schindler's notes [8]. Assume for a contradiction that NS_{ω_1} is not $\aleph_2$-saturated in $V[G]$, i.e., there is a maximal antichain $\vec{S} = (S_i : i < \omega_2)$ in $P(\omega_1)/\mathrm{NS}_{\omega_1}$. Let τ be a $\mathbb{P}$-name for the sequence. As $V[G] \models \aleph_2 = \Lambda$ for our Woodin cardinal Λ, we claim that it is possible to find an inaccessible κ below Λ such that the following three properties hold:

1. κ is $\mathbb{P} \oplus \tau$-strong up to Λ in V,
2. $\kappa = \omega_2^{V[G \restriction \kappa]}$,
3. $\vec{S} \restriction \kappa = (S_i : i < \kappa) = (\tau \cap V_\kappa)^{G \restriction \kappa}$ is the maximal antichain in $V[G \restriction \kappa]$ which is picked by the $\diamondsuit$-sequence at stage κ.

This is clear as we can assume that our $\diamondsuit$-sequence lives on the stationary subset of inaccessible cardinals below δ, and for all inaccessible κ property 2 automatically holds. Moreover the sets

$$C_1 := \{\kappa < \delta \, : \, \vec{S} \restriction \kappa = (S_i \, : \, i < \kappa) = (\tau \cap V_\kappa)^{V[G \restriction \kappa]}\}$$

and

$$C_2 := \{\kappa < \delta \, : \forall \alpha < \kappa \forall S \in P(\omega_1) \cap V^{\mathbb{P}_\alpha} \text{ stationary } \exists \bar{S} \in \vec{S} \restriction \kappa (S \cap \bar{S} \notin NS)\}$$

are both clubs, therefore hitting the stationary set T consisting of the points $\kappa < \Lambda$ where $\tau \cap V_\kappa = a_\kappa$ and κ is τ-strong up to Λ. Thus, if κ is in the nonempty intersection $C_1 \cap C_2 \cap T$ then 1 and 2 are satisfied, and the recursive definition of our forcing $\mathbb{P}$ yields that at stage κ, as $a_\kappa = \tau \cap V_\kappa$, the sealing forcing $\mathbb{S}((\tau \cap V_\kappa)^{G \restriction \kappa})$ is at least considered, and in order to show property 3, it suffices to show that $(\tau \cap V_\kappa)^{G \restriction \kappa}) = \bar{S} \restriction \kappa$ is maximal in $V[G \restriction \kappa]$. But this is clear as by the definition of RCS iteration and as $|\mathbb{P}_\alpha| < \kappa$ we take at inaccessible κ's the direct limit of the $\mathbb{P}_\alpha$'s, thus each stationary $S \subset \omega_1$ in $V^{\mathbb{P}_\kappa}$ is already included in a $V^{\mathbb{P}_\alpha}$ for $\alpha < \kappa$. So we have ensured the existence of a κ with all the 3, above stated properties.

Now the forcing $\mathbb{S}(\vec{S} \upharpoonright \kappa)$ can not be semiproper at stage κ, as otherwise we would have to force with it, therefore killing the antichain $\vec{S}$. So there exists a condition $(p, c) \in \mathbb{S}(\vec{S} \upharpoonright \kappa)$ such that the set

$$\bar{T} := \{X \prec (H_{\kappa^+})^{V[G\upharpoonright\kappa]} : |X| = \aleph_0 \wedge (p, c) \in X \wedge \nexists Y \supset X (Y \prec (H_{\kappa^+})^{V[G\upharpoonright\kappa]} \\ \wedge |Y| = \aleph_0 \wedge (X \cap \omega_1 = Y \cap \omega_1) \wedge \exists (q, d) \leq (p, c) ((q, d) \text{ is Y-semigeneric }))\}.$$

is stationary in $V[G \upharpoonright \kappa]$, and by construction of our iteration, the κ-th forcing in $\mathbb{P}$ is $Col(\omega_1, 2^{\aleph_2})$, so in $V[G \upharpoonright \kappa + 1]$ there is a surjection $f : \omega_1 \to (H_{\kappa^+})^{V[G\upharpoonright\kappa]}$. As $Col(\omega_1, 2^{\aleph_2})$ is proper the set $\bar{T}$ remains stationary in $V[G \upharpoonright \kappa + 1]$ which implies that

$$T := \{\alpha < \omega_1 : f"\alpha \in \bar{T} \wedge \alpha = f"\alpha \cap \omega_1\}$$

is stationary in $V[G \upharpoonright \kappa + 1]$. As the tail $\mathbb{P}_{[\kappa+2,\Lambda)}$ remains semiproper, seen as an iteration with $V[G \upharpoonright \kappa + 1]$ as ground model, we can infer that T remains stationary in $V[G]$ and hence there exists an $i_0 < \Lambda$ such that

$$(**) \quad T \cap S_{i_0} \text{ is stationary in } V[G].$$

Let us shortly reflect the situation we are in. The idea is to find a model $X \in \bar{T}$ such that we *can* find a $(X, \mathbb{S}(\vec{S} \upharpoonright \kappa))$-semigeneric condition $(q, d) < (p, c)$, thus arriving at a contradiction. In order to do so we have to ensure that $\alpha = X \cap \omega_1$ is in some $S_i \in \vec{S} \upharpoonright \kappa$. As $\vec{S}$ was assumed to be maximal there is indeed an index $i_0 < \delta$ which is as desired, this index however might be bigger than κ. This is where the large cardinal assumption comes into play. We can find an elementary embedding $j : V \to M$ which fixes the name for the antichain $\vec{S}$ and such that $j(\kappa) > i_0$. We shall use a lifted version of this elementary embedding j to derive a contradiction.

First, let $\lambda < \Lambda$, $\lambda > \max(i_0, \kappa + 1)$ be such that $(\tau \cap V_\lambda)^{G\upharpoonright\lambda} = \vec{S} \upharpoonright \lambda$, so we have $(\tau \cap V_\lambda)^{G\upharpoonright\lambda}(i_0) = S_{i_0}$. As κ was chosen to be $\mathbb{P} \oplus \tau$-strong up to Λ we let $j : V \to M$ be an elementary embedding with critical point κ, such that M is transitive, $M^\kappa \subset M$, $V_{\lambda+\omega} \subset M$, $j(\mathbb{P}) \cap V_\lambda = \mathbb{P} \cap V_\lambda$, and $j(\tau) \cap V_\lambda = \tau \cap V_\lambda$.

H should denote the generic filter for the segment $(\mathbb{P}_{[\lambda+1,j(\kappa)]})^{M[G\upharpoonright\lambda]}$ of $j(\mathbb{P})$ over $M[G \upharpoonright \lambda]$. Then, we lift j to an elementary embedding

$$j^* : V[G \upharpoonright \kappa] \to M[G \upharpoonright \lambda, H].$$

Notice that $(V_{\lambda+\omega})^{V[G\upharpoonright\lambda]} = (V_{\lambda+\omega})^{M[G\upharpoonright\lambda]}$.

Now we let $(X_i : i < \omega_1) \in V[G \upharpoonright \kappa + 1]$ be an increasing continuous chain of countable elementary substructures of $(H_{j(\kappa)^+})^{M[G\upharpoonright\kappa+1]}$ with $\{\tau \cap V_\lambda, i_0\} \subset X_0$ satisfying for all $i < \omega_1$ the following three properties:

(a) $i \in X_{i+1}$

(b) $f"(X_i \cap \omega_1) \subset X_i$

(c) $j"(X_i \cap (H_{\kappa^+})^{V[\mathbb{P}_\kappa]}) \subset X_i$

Let $\bar{G} := G \upharpoonright [\kappa+2, \lambda]$, then we have that

$$\{X_i[\bar{G}] \cap \omega_1 \,:\, i < \omega_1\} \in V[G \upharpoonright \lambda]$$

is a club in ω_1 so intersecting it with the stationary set defined in $(**)$ we find some $i < \omega_1$ such that $X_i[\bar{G}] \cap \omega_1 = X_i \cap \omega_1 \in T \cap S_{i_0}$.

Write $X := X_i, \alpha := X \cap \omega_1$. As at stage κ we had to force with the ω-closed $Col(2^{\aleph_2}, \aleph_1)$ we know that $X \cap (H_{\kappa^+})^{V[G\upharpoonright\kappa]} \in V[G \upharpoonright \kappa]$. Remember that $f \in V[G \upharpoonright \kappa+1]$ was chosen as a surjection of ω_1 onto $(H_{\kappa^+})^{V[G\upharpoonright\kappa]}$, so as $\alpha \in T$ by definition of T $f"\alpha \in \bar{T}$ and $\alpha = f"\alpha \cap \omega_1$, and hence by (b)

$$f"\alpha \subset X \cap (H_{\kappa^+})^{V[G\upharpoonright\kappa]} \in V[G \upharpoonright \kappa].$$

As $\alpha = f"\alpha \cap \omega_1$, $f"\alpha \in \bar{T}$ and $f"\alpha \subset X \cap (H_{\kappa^+})^{V[G\upharpoonright\kappa]}$ we get that $X \cap (H_{\kappa^+})^{V[G\upharpoonright\kappa]} \in \bar{T}$ and therefore

$$(***) \quad j^*(X \cap (H_{\kappa^+})^{V[G\upharpoonright\kappa]}) \in j^*(\bar{T}).$$

Note that our second generic H, denoting the generic filter for the segment $(\mathbb{P}_{[\lambda+1,j(\kappa)]})^{M[G\upharpoonright\lambda]}$ of $j(\mathbb{P})$ over $M[G \upharpoonright \lambda]$ has not been specified yet. As the segment $(\mathbb{P}_{[\lambda+1,j(\kappa)]})^{M[G\upharpoonright\lambda]}$ of $j(\mathbb{P})$ over $M[G \upharpoonright \lambda]$ is semi-proper we have that there is a condition q in the segment $(\mathbb{P}_{[\lambda+1,j(\kappa)]})^{M[G\upharpoonright\lambda]}$ of $j(\mathbb{P})$ which is $(X[\bar{G}], \mathbb{P}_{[\lambda+1,j(\kappa)]})$-semigeneric. If we pick H such that $q \in H$, then by semigenericity of q, we obtain $X[\bar{G}, H] \cap \omega_1 = X[\bar{G}] \cap \omega_1 = X \cap \omega_1 = \alpha \in S_{i_0} = (\tau \cap V_\lambda)^{G\upharpoonright\lambda}(i_0) \in X[\bar{G}, H]$. But also due to (c) we have that

$$j^*(X \cap (H_{\kappa^+})^{V[G\upharpoonright\kappa]}) = j^*"(X \cap (H_{\kappa^+})^{V[G\upharpoonright\kappa]}) \subset X[\bar{G}, H].$$

This gives us the desired contradiction as we can find an $(X[\bar{G}, H], j(\mathbb{S}(\vec{S} \upharpoonright \kappa)))$-generic condition below $j(p,c) = (p,c)$. Indeed, we can just list the countably many names for countable ordinals in $X[\bar{G}, H]$ along with conditions of $j(\mathbb{S}(\vec{S} \upharpoonright \kappa))$ deciding them below (p,c) and let $(p', c') \in j(\mathbb{S}(\vec{S} \upharpoonright \kappa))$ be just the lower bound of that sequence, i.e, the condition with $\mathrm{dom}(c') = \alpha+1$, $c'(\alpha) = \alpha$ and $p'(i) = S_{i_0}$ for some $i < \alpha$. Note here that we can assume that (p', c') is also an element of $V[G \upharpoonright \kappa]$, as we can assume that the extender which gives rise to the elementary embedding $j : V \to M$ is κ-closed. So $X[\bar{G}, H]$ together with the fact that $X[\bar{G}, H] \cap \omega_1 = \alpha = j^*(X \cap H_{\kappa^+}^{V[G\upharpoonright\kappa]}) \cap \omega_1$ witnesses that the condition $(p', c') < (p, c)$ is $(j^*(X \cap (H_{\kappa^+})^{V[G\upharpoonright\kappa]}), j^*(\mathbb{S}(\vec{S} \upharpoonright \kappa))$-semigeneric, hence $j^*(X \cap (H_{\kappa^+})^{V[G\upharpoonright\kappa]}) \notin j^*(\bar{T})$, contradicting $(***)$. $\square$

We end with a short remark and an open question. The natural follow up to ask is whether the Σ^1_4-wellorder can be improved to a Σ^1_3-wellorder? This question is tied to the notorious problem of whether NS_{ω_1} and CH are consistent, as by the already mentioned result of G. Hjorth (see [4]), a Σ^1_3-definable wellorder in the presence of "every real has a sharp" implies CH. Thus, there could be a possibility of an even better projective wellorder of the reals and its existence could settle $Con(\mathrm{NS}_{\omega_1}$ is saturated + $\mathsf{CH})$. Of course this can happen only in a model with no measurable cardinal by Woodin's result.

A second interesting problem is the question of the definability of NS_{ω_1} over the structure $H(\omega_2)$ if we additionally demand NS_{ω_1} to be saturated. Woodin has shown that from ω many Woodin cardinals one obtains a model in which NS_{ω_1} is ω_1-dense (i.e., $RO(P(\omega_1)/\mathrm{NS}_{\omega_1})$ has an $\aleph_1$-sized, dense subfamily) , which implies its saturation and Δ_1-definability of stationarity using the dense family as a parameter. In [2] it is asked whether the large cardinal assumptions can be lowered. That this is indeed the case has been shown recently by the second author, who showed that given a Woodin cardinal there is a model of ZFC where NS_{ω_1} is saturated and $\Delta_1(\omega_1)$-definable.

3 Acknowledgements

The results in this article form a part of the second authors Ph.D. thesis, supervised by the first author. Both would like to thank the Austrian Science Fund FWF for its generous support through research project P25748. The second author was additionally funded by FWF project I1272

References

[1] A. Caicedo and B. Velickovic *The bounded proper forcing axiom and wellorderings of the reals.* Mathematical Research Letters, vol 13 (2-3) (2006), no.2-3, pp. 393-408.

[2] S. D. Friedman and L. Wu, *Large cardinals and the Δ_1-definability of the nonstationary ideal.* to appear.

[3] M. Gitik and S. Shelah, *Less saturated ideals.* Proceedings of the American Mathematical Society vol. 125 (1997), No. 5, pp. 1523-1530

[4] G. Hjorth, *The size of the ordinal u_2* Journal of the London Mathematical Society (2), vol 52 (1995), no. 5, pp. 417-433.

[5] R. Jensen and J. Steel, *K without a measurable.* this Journal, vol 78 (2013), no 3, pp. 708-734.

[6] K. Kunen, *Saturated Ideals.* this Journal, vol. 43 (1978), no. 1, pp. 65-76

[7] C. Schlindwein, *Simplified RCS iterations.* Archive for Mathematical Logic, vol 32 (1993), no 5, pp. 341-349.

[8] R. Schindler, *On NS_{ω_1} being saturated.* Online Notes, 2016 Available at `http://www.math.uni-muenster.de/u/rds/sat_ideal_better_version.pdf`.

[9] J. Steel, *Inner models with many Woodin cardinals.* Annals of Pure and applied Logic, vol 65 (1993), pp.185-209.

[10] J. Steel, *Projectively wellordered inner models.* Annals of pure and applied Logic, (1995), pp.77-104.

[11] H. W. Woodin, *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal,* De Gruyter, 2001.

KURT GÖDEL RESEARCH CENTER
UNIVERSITÄT WIEN
E-mail: sdf@logic.univie.ac.at

WESTFÄLISCHE WILHELMSUNIVERSITÄT MÜNSTER
E-mail: hoffelne@uni-muenster.de